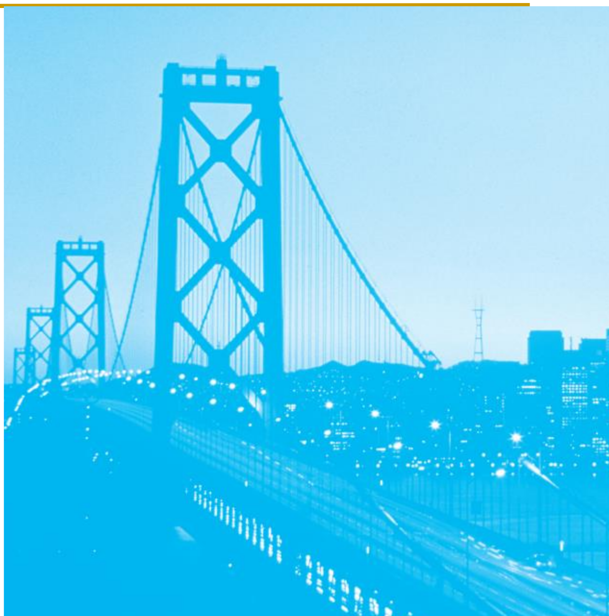
Chapter 4

Systems of ODEs. Phase Plane. Qualitative Methods



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Contents

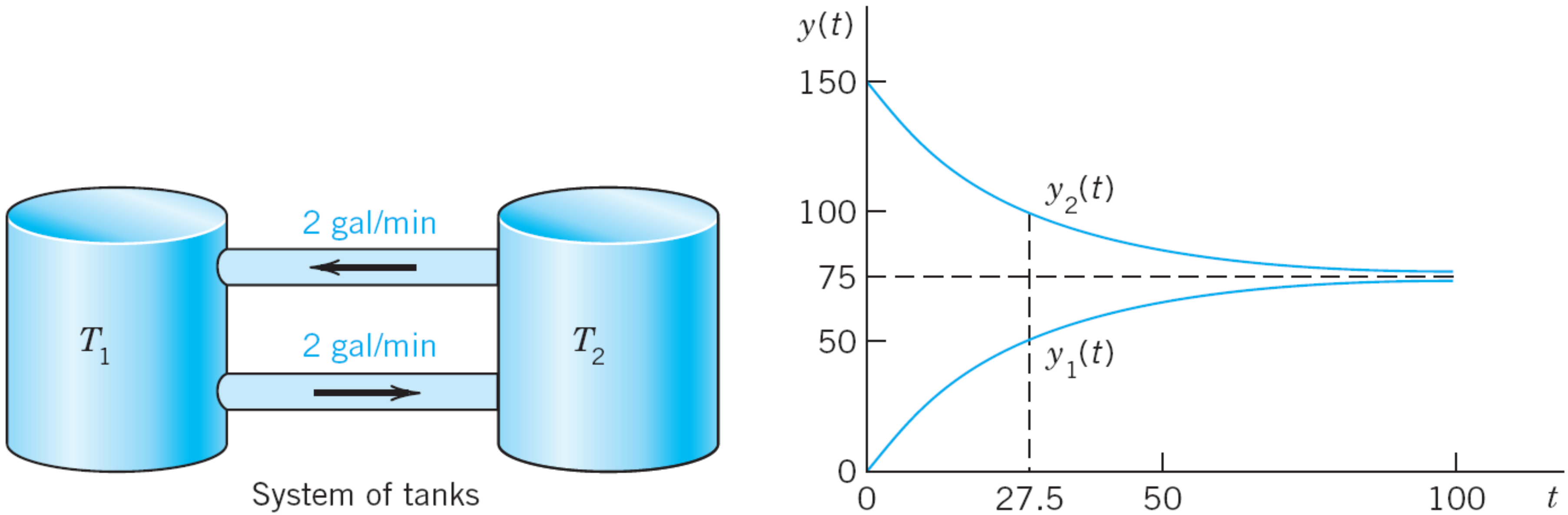
4.1 Systems of ODEs as Models

Example 1 Mixing Problem Involving Two Tanks

- A mixing problem involving a single tank can be modeled by a single ODE. The model will be a system of two first-order ODEs.
- Tank T_1 and T_2 in Fig. 77 contain initially 100 gal of water each. In T_1 the water is pure, whereas 150 lb of fertilizer are dissolved in T_2. By circulating liquid at a rate of 2 gal/min and stirring (to keep the mixture uniform) the amounts of fertilizer $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t.
- How long should we let the liquid circulate so that T_1 will contain at least half as much fertilizer as there will be left in T_2?

continued

Fig. 77. Fertilizer content in Tanks *T*1 (lower curve) and *T*2



continued

Solution. Step 1. **Setting up the model.** As for a single tank, the time rate of change $y'_1(t)$ of $y_1(t)$ equals inflow minus outflow. Similarly for tank T_2. From Fig. 77 we see that

$$y'_1 = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100} y_2 - \frac{2}{100} y_1 \qquad \text{(Tank } T_1\text{)}$$

$$y'_2 = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100} y_1 - \frac{2}{100} y_2 \qquad \text{(Tank } T_2\text{).}$$

Hence the mathematical model of our mixture problem is the system of first-order ODEs

$$y'_1 = -0.02y_1 + 0.02y_2 \qquad \text{(Tank } T_1\text{)}$$

$$y'_2 = 0.02y_1 - 0.02y_2 \qquad \text{(Tank } T_2\text{).}$$

continued

As a vector equation with column vector $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and matrix **A** this becomes

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}.$$

Step 2. **General solution.**

As for a single equation, we try an exponential function of t,

(1) $\mathbf{y} = \mathbf{x}e^{\lambda t}$. Then $\mathbf{y}' = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$.

Dividing the last equation $\lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$ by $e^{\lambda t}$ and interchanging the left and right sides, we obtain

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$$

continued

Since nontrivial solutions are needed, we have to look for eigenvalues and eigenvectors of **A.** The eigenvalues are the solutions of the characteristic equation

(2)

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -0.02 - \lambda & 0.02 \\ 0.02 & -0.02 - \lambda \end{vmatrix} = (-0.02 - \lambda)^2 - 0.02^2 = \lambda(\lambda + 0.04) = 0.$$

We see that $\lambda_1 = 0$ and $\lambda_2 = -0.04$. For our present **A** this gives

a) $-0.02x_1 + 0.02x_2 = 0$ $(\lambda_1 = 0)$

b) $(-0.02 + 0.04)x_1 + 0.02x_2 = 0$ $(\lambda_2 = -0.04)$

continued

Hence $x_1 = x_2$ and $x_1 = -x_2$, respectively, and we can take $x_1 = x_2 = 1$ and $x_1 = -x_2 = 1$. This gives two eigenvectors corresponding to $\lambda_1 = 0$ and $\lambda_2 = -0.04$, respectively, namely,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

From (1) and the superposition principle, we thus obtain a solution

(3) $$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

where c_1 and c_2 are arbitrary constants. Later we shall call this a **general solution**.

continued

***Step 3.* Use of initial conditions.** The initial conditions are $y_1(0) = 0$ (no fertilizer in tank T_1) and $y_2(0) = 150$. From this and (3) with $t = 0$ we obtain

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}.$$

In components this is $c_1 + c_2 = 0$, $c_1 - c_2 = 150$. The solution is $c_1 = 75$, $c_2 = -75$. This gives the answer

$$\mathbf{y} = 75\mathbf{x}^{(1)} - 75\mathbf{x}^{(2)}e^{-0.04t} = 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}.$$

$$\therefore \quad y_1 = 75 - 75e^{-0.04t}; \quad y_2 = 75 + 75e^{-0.04t}$$

Figure 77 shows the exponential increase of y1 and the exponential decrease of y2 to the common limit 75 lb.

continued

Step 4.

Answer. T_1 contains half the fertilizer amount of T_2 if it contains 1/3 of the total amount, that is, 50 lb. Thus

$$y_1 = 75 - 75e^{-0.04t} = 50, \qquad e^{-0.04t} = 1/3,$$

$$t = (\ln 3)/0.04 = 27.5.$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y}; \qquad \mathbf{y}(0) = \mathbf{y}_0$$

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} \Rightarrow \mathbf{X}\mathbf{D}\mathbf{X}^{-1} = \mathbf{A}$$

$$\mathbf{y}' = \mathbf{X}\mathbf{D}\mathbf{X}^{-1}\mathbf{y}$$

$$\mathbf{X}^{-1}\mathbf{y}' = \mathbf{D}\mathbf{X}^{-1}\mathbf{y}$$

$$\text{Let } \mathbf{z} = \mathbf{X}^{-1}\mathbf{y} \quad \Rightarrow \quad \mathbf{z}' = \mathbf{D}\mathbf{z}$$

$$\begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$z_1 = c_1 e^{\lambda_1 t}, z_2 = c_2 e^{\lambda_2 t}, \cdots, z_n = c_n e^{\lambda_n t}$$

$$\mathbf{y} = \mathbf{X}\mathbf{z} = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \cdots & \mathbf{x}^{(n)} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \sum_{k=1}^{n} c_k \mathbf{x}^{(k)} e^{\lambda_k t}$$

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}\mathbf{r}(t); \qquad \mathbf{y}(0) = \mathbf{y}_0$$

$\mathbf{g}(t)$

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} \Rightarrow \mathbf{X}\mathbf{D}\mathbf{X}^{-1} = \mathbf{A}$$

$$\mathbf{y}' = \mathbf{X}\mathbf{D}\mathbf{X}^{-1}\mathbf{y} + \mathbf{g}$$

$$\mathbf{X}^{-1}\mathbf{y}' = \mathbf{D}\mathbf{X}^{-1}\mathbf{y} + \mathbf{X}^{-1}\mathbf{g}$$

$$\text{Let } \mathbf{z} = \mathbf{X}^{-1}\mathbf{y} \text{ and } \mathbf{h} = \mathbf{X}^{-1}\mathbf{g}$$

$$\Rightarrow \quad \mathbf{z}' = \mathbf{D}\mathbf{z} + \mathbf{h}$$

$$\begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{bmatrix}$$

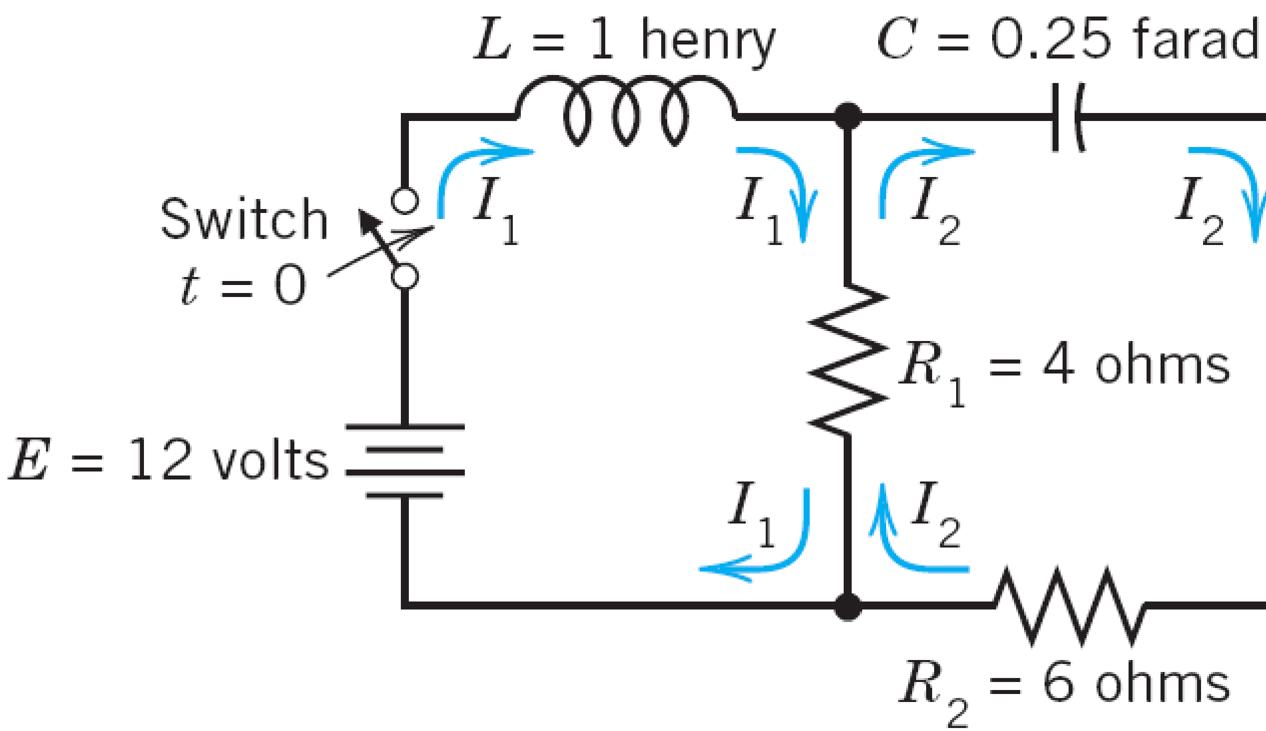
$$z_k = c_k e^{\lambda_k t} + e^{\lambda_k t} \int h_k(\tau) e^{-\lambda_k \tau} d\tau; \qquad k = 1, 2, \cdots, n$$

$$\mathbf{y} = \mathbf{X}\mathbf{z} = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \cdots & \mathbf{x}^{(n)} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \sum_{k=1}^{n} z_k \mathbf{x}^{(k)}$$

$$= \underbrace{\sum_{k=1}^{n} c_k \mathbf{x}^{(k)} e^{\lambda_k t}}_{\mathbf{y}_h} + \underbrace{\sum_{k=1}^{n} \mathbf{x}^{(k)} e^{\lambda_k t} \int h_k(\tau) e^{-\lambda_k \tau} d\tau}_{\mathbf{y}_p}$$

Example 2 Electrical Network

Find the currents $I_1(t)$ and $I_2(t)$ in the network as shown. Assume all currents and charges to be zero at $t = 0$, the instant when the switch is closed.



continued

Solution. ***Step 1.*** **Setting up the mathematical model.** The model of this network is obtained from Kirchhoff's voltage law, as in Sec. 2.9 (where we considered single circuits). Let $I_1(t)$ and $I_2(t)$ be the currents in the left and right loops, respectively. In the left loop the voltage drops are $LI'_1 = I'_1$ [V] over the inductor and $R_1(I_1 - I_2) = 4(I_1 - I_2)$ [V] over the resistor, the difference because I_1 and I_2 flow through the resistor in opposite directions. By Kirchhoff's voltage law the sum of these drops equals the voltage of the battery; that is, $I'_1 + 4(I_1 - I_2) = 12$, hence

$$I'_1 = -4I_1 + 4I_2 + 12. \qquad (4a)$$

continued

In the right loop the voltage drops are $R_2 I_2 = 6I_2$ [V] and $R_1(I_2 - I_1) = 4(I_2 - I_1)$ [V] over the resistors and

$(1/C)\int I_2\, dt = 4\int I_2\, dt$ [V] over the capacitor, and their sum is zero,

$$6I_2 + 4(I_2 - I_1) + 4\int I_2\, dt = 0 \qquad \text{or} \qquad 10I_2 - 4I_1 + 4\int I_2\, dt = 0.$$

Division by 10 and differentiation gives $I'_2 - 0.4I'_1 + 0.4I_2 = 0$.

To simplify the solution process, we first get rid of $0.4I'_1$, which by (4a) equals $0.4(-4I_1 + 4I_2 + 12)$. Substitution into the present ODE gives

$$I'_2 = 0.4I'_1 - 0.4I_2 = 0.4(-4I_1 + 4I_2 + 12) - 0.4I_2$$

continued

and by simplification

(4b) $$I'_2 = -1.6I_1 + 1.2I_2 + 4.8.$$

In matrix form, (4) is (we write **J** since **I** is the unit matrix)

(5) $$\mathbf{J}' = \mathbf{A}\mathbf{J} + \mathbf{g}, \quad \text{where} \quad \mathbf{J} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 12.0 \\ 4.8 \end{bmatrix}.$$

***Step 2.* Solving (5).** Because of the vector **g** this is a *nonhomogeneous* system.

We try to proceed as for a single ODE, solving first the *homogeneous* system $\mathbf{J}' = \mathbf{A}\mathbf{J}$ by substituting $\mathbf{J} = \mathbf{x}e^{\lambda t}$

$$\mathbf{J}' = \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$$

$$\therefore \mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

continued

Hence to obtain a nontrivial solution, we again need the eigenvalues and eigenvectors:

$$\lambda_1 = -2, \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \qquad \lambda_2 = -0.8, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}.$$

Hence a "general solution" of the homogeneous system is

$$\mathbf{J}_h = c_1\mathbf{x}^{(1)}e^{-2t} + c_2\mathbf{x}^{(2)}e^{-0.8t}.$$

For a particular solution of the nonhomogeneous system (5), since **g** is constant, we try a constant column vector $\mathbf{J}_p = \mathbf{a}$ with components a_1, a_2.

Then $\mathbf{J}'_p = \mathbf{0}$, and substitution into (5) gives

$$\mathbf{Aa} + \mathbf{g} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 12.0 \\ 4.8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

continued

$$-4.0a_1 + 4.0a_2 + 12.0 = 0$$

$$-1.6a_1 + 1.2a_2 + 4.8 = 0.$$

The solution is $a_1 = 3$, $a_2 = 0$; thus $\mathbf{a} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Hence

(6) $$\mathbf{J} = \mathbf{J}_h + \mathbf{J}_p = c_1\mathbf{x}^{(1)}e^{-2t} + c_2\mathbf{x}^{(2)}e^{-0.8t} + \mathbf{a};$$

$$= c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ 0.8 \end{bmatrix}e^{-0.8t} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

in components,

$$I_1 = 2c_1e^{-2t} + c_2e^{-0.8t} + 3$$

$$I_2 = c_1e^{-2t} + 0.8c_2e^{-0.8t}.$$

continued

The initial conditions give

$$I_1(0) = 2c_1 + c_2 + 3 = 0$$

$$I_2(0) = c_1 + 0.8c_2 = 0.$$

Hence $c_1 = -4$ and $c_2 = 5$. As the solution of our problem we thus obtain

$$\text{(7)} \qquad \mathbf{J} = -4\mathbf{x}^{(1)}e^{-2t} + 5\mathbf{x}^{(2)}e^{-0.8t} + \mathbf{a}.$$

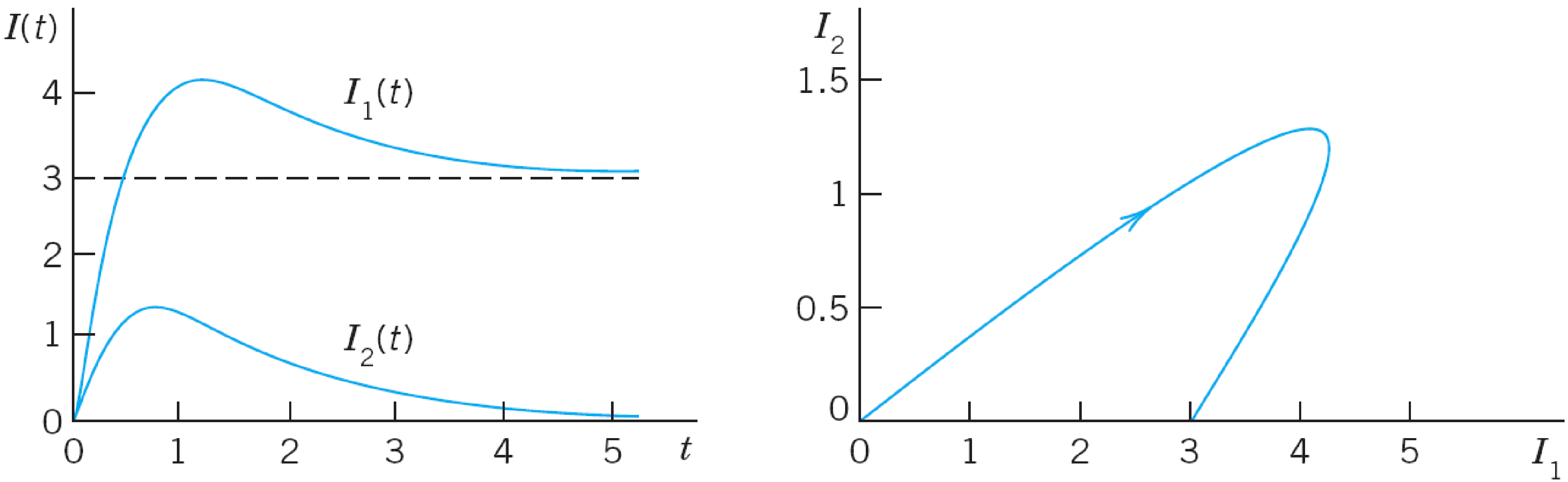
In components (Fig. 79b),

$$I_1 = -8e^{-2t} + 5e^{-0.8t} + 3$$

$$I_2 = -4e^{-2t} + 4e^{-0.8t}.$$

continued

Fig. 79. Currents in Example 2



(a) Currents I_1 (upper curve) and I_2

(b) Trajectory $[I_1(t), I_2(t)]^T$ in the I_1I_2-plane (the "phase plane")

Conversion of an *n*th-Order ODE to a System

THEOREM 1

Conversion of an ODE

An nth-order ODE

(8) $$y^{(n)} = F\left(t, y, y', \cdots, y^{(n-1)}\right)$$

can be converted to a system of n first-order ODEs by setting

(9) $$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \quad \cdots , \quad y_n = y^{(n-1)}.$$

This system is of the form

(10)
$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \cdots, y_n). \end{aligned}$$

Example 3 Mass on a Spring

Let us consider the free motions of a mass on a spring

$$my'' + cy' + ky = 0 \qquad \text{or} \qquad y'' = -\frac{c}{m}y' - \frac{k}{m}y.$$

For this ODE (8) the system (10) is linear and homogeneous,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{k}{m}\,y_1 - \frac{c}{m}\,y_2. \end{aligned}$$

continued

Setting $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ we get in matrix form

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

For an illustrative computation, let $m = 1$, $c = 2$, and $k = 0.75$:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + 0.5)(\lambda + 1.5) = 0.$$

This gives the eigenvalues $\lambda_1 = -0.5$ and $\lambda_2 = -1.5$.

continued

Eigenvectors follow from the first equation in $\mathbf{A} - \lambda\mathbf{I} = \mathbf{0}$, which is $-\lambda x_1 + x_2 = 0$.

For λ_1 this gives $0.5x_1 + x_2 = 0$, say, $x_1 = 2$, $x_2 = -1$.

For $\lambda_2 = -1.5$ it gives $1.5x_1 + x_2 = 0$, say, $x_1 = 1$, $x_2 = -1.5$. These eigenvectors

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} \qquad \text{give} \qquad \mathbf{y} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}.$$

This vector solution has the first component

$$y = y_1 = 2c_1e^{-0.5t} + c_2e^{-1.5t}$$

which is the expected solution. The second component is its derivative

$$y_2 = y'_1 = y' = -c_1e^{-0.5t} - 1.5c_2e^{-1.5t}.$$

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + a_2 \frac{d^{n-2} y(t)}{dt^{n-2}} + \cdots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = r(t)$$

$$y_1(t) = y(t)$$

$$y_2(t) = \frac{dy(t)}{dt} = \frac{dy_1(t)}{dt}$$

$$y_3(t) = \frac{d^2 y(t)}{dt^2} = \frac{d}{dt}\frac{dy(t)}{dt} = \frac{dy_2(t)}{dt}$$

$$\vdots$$

$$y_{n-1}(t) = \frac{d^{n-2} y(t)}{dt^{n-2}} = \frac{dy_{n-2}(t)}{dt}$$

$$y_n(t) = \frac{d^{n-1} y(t)}{dt^{n-1}} = \frac{dy_{n-1}(t)}{dt}$$

$$-a_n y_1(t) - a_{n-1} y_2(t) - \cdots - a_2 y_{n-1}(t) - a_1 y_n(t) + r(t) = \frac{dy_n(t)}{dt}$$

Phase Variable Canonical Form

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t) + \mathbf{B}r(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & -a_{n-4} & \cdots & -a_1 \end{bmatrix}_{n\times n}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}$$

Vandermonde Matrix

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

$$\text{let} \quad \mathbf{x}^{(i)} = \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \\ \vdots \\ x_n^{(i)} \end{bmatrix} \quad \Rightarrow \quad (\lambda_i \mathbf{I} - \mathbf{A})\mathbf{x}^{(i)} = \mathbf{0}$$

$$\begin{bmatrix} \lambda_i & -1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & -1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \lambda_i & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & \lambda_i + a_1 \end{bmatrix} \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \\ x_3^{(i)} \\ \vdots \\ x_{n-1}^{(i)} \\ x_n^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_i x_1^{(i)} - x_2^{(i)} = 0$$

$$\lambda_i x_2^{(i)} - x_3^{(i)} = 0$$

$$\vdots$$

$$\lambda_i x_{n-1}^{(i)} - x_n^{(i)} = 0$$

$$a_n x_1^{(i)} + a_{n-1} x_2^{(i)} + \cdots + a_2 x_{n-1}^{(i)} + \left(\lambda_i + a_1\right) x_n^{(i)} = 0$$

Let $x_1^{(i)} = 1$

$$\Rightarrow x_2^{(i)} = \lambda_i; \quad x_3^{(i)} = \lambda_i^2; \quad x_4^{(i)} = \lambda_i^3; \quad \cdots \quad ; \quad x_{n-1}^{(i)} = \lambda_i^{n-2}; \quad x_n^{(i)} = \lambda_i^{n-1}$$

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$$

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

4.3 Constant-Coefficient Systems. Phase Plane Method

THEOREM 1

General Solution

If the constant matrix A in the system (1) has a linearly independent set of n eigenvectors, then the corresponding solutions $y^{(1)}, \ldots, y^{(n)}$ *in (4) form a basis of solutions of (1), and the corresponding general solution is*

$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + \cdots + c_n \mathbf{x}^{(n)} e^{\lambda_n t}. \tag{5}$$

How to Graph Solutions in the Phase Plane

- We shall now concentrate on systems (1) with constant coefficients consisting of two ODEs

(6) $\mathbf{y}' = \mathbf{A}\mathbf{y};$ in components, $$y'_1 = a_{11}y_1 + a_{12}y_2$$ $$y'_2 = a_{21}y_1 + a_{22}y\,.$$

Of course, we can graph solutions of (6),

(7) $$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix},$$

continued

as two curves over the t-axis, one for each component of $\mathbf{y}(t)$. (Figure 79a in Sec. 4.1 shows an example.) But we can also graph (7) as a single curve in the y_1y_2-plane. This is a *parametric representation* (*parametric equation*) with parameter t. (See Fig. 79b for an example. Many more follow. Parametric equations also occur in calculus.) Such a curve is called a **trajectory** (or sometimes an *orbit* or *path*) of (6). The y_1y_2-plane is called the **phase plane.** If we fill the phase plane with trajectories of (6), we obtain the so-called **phase portrait** of (6).

EXAMPLE 1 Trajectories in the Phase Plane (Phase Portrait)

- In order to see what is going on, let us find and graph solutions of the system

(8) $$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}, \quad \text{thus} \quad \begin{aligned} y_1' &= -3y_1 + y_2 \\ y_2' &= y_1 - 3y_2. \end{aligned}$$

- ***Solution.*** By substituting $\mathbf{y} = \mathbf{x}e^{\lambda t}$ and $\mathbf{y}' = \mathbf{x}e^{\lambda t}$ and dropping the exponential function we get $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = 0.$$

This gives the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$. Eigenvectors are then obtained from

$$(-3 - \lambda)x_1 + x_2 = 0.$$

continued

For $\lambda_1 = -2$ this is $-x_1 + x_2 = 0$. Hence we can take $\mathbf{x}^{(1)} = [1 \quad 1]^T$. For $\lambda_2 = -4$ this becomes $x_1 + x_2 = 0$, and an eigenvector is $\mathbf{x}^{(2)} = [1 \quad -1]^T$. This gives the general solution

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}.$$

- Figure 81 shows a phase portrait of some of the trajectories (to which more trajectories could be added if so desired). The two straight trajectories correspond to $c_1 = 0$ and $c_2 = 0$ and the others to other choices of c_1, c_2.

Critical Points of the System (6)

- The point **y** = **0** in Fig. 81 seems to be a common point of all trajectories, and we want to explore the reason for this remarkable observation. The answer will follow by calculus. Indeed, from (6) we obtain

(9)
$$\frac{dy_2}{dy_1} = \frac{y_2'\,dt}{y_1'\,dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}.$$

This associates with every point P: (y_1, y_2) a unique tangent direction dy_2/dy_1 of the trajectory passing through P, except for the point $P = P_0$: (0, 0), where the right side of (9) becomes 0/0. This point P_0, at which dy_2/dy_1 becomes undetermined, is called a **critical point** of (6).

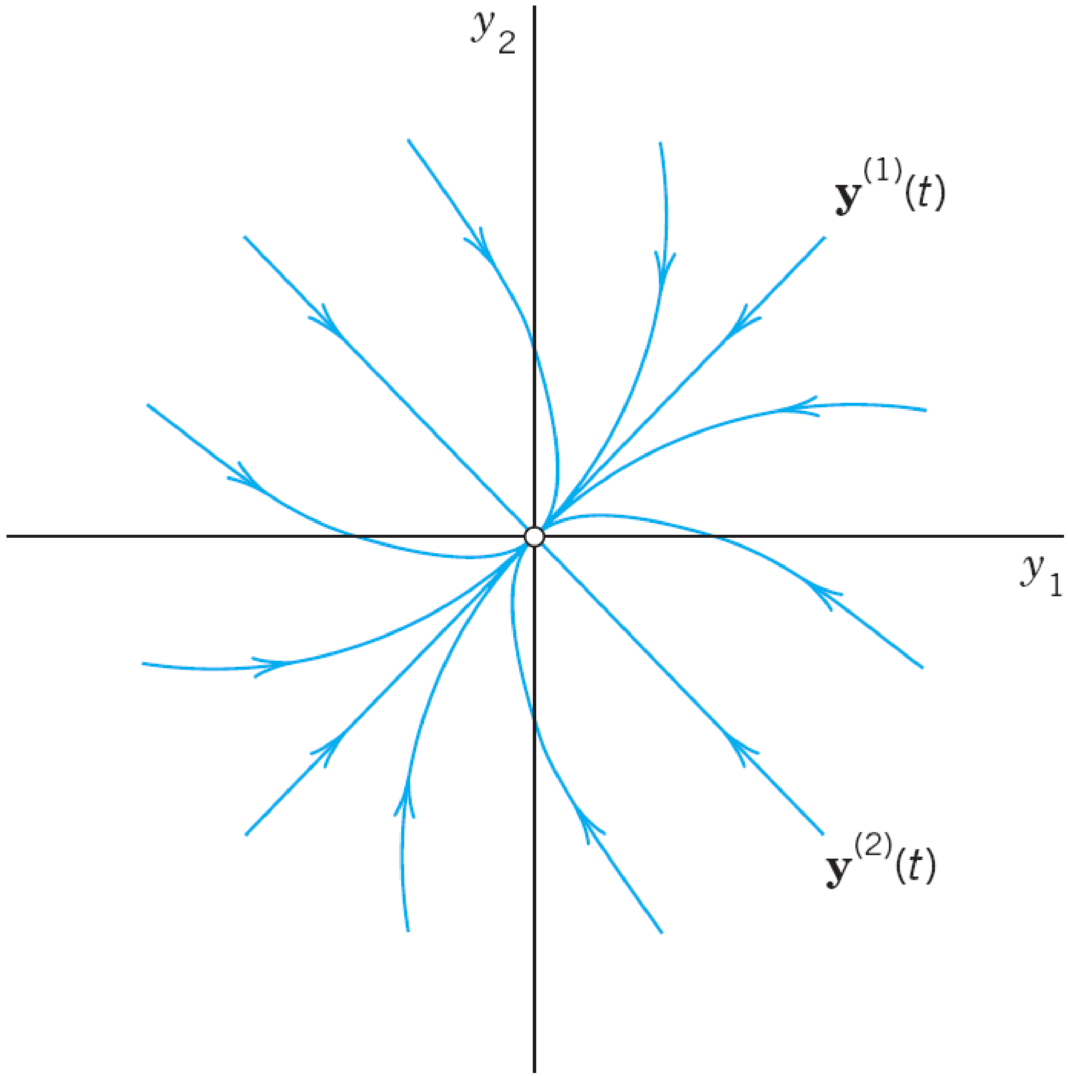
Five Types of Critical Points

EXAMPLE 1 (*Continued*) Improper Node (Fig. 81)

- An **improper node** is a critical point P_0 at which all the trajectories, except for two of them, have the same limiting direction of the tangent. The two exceptional trajectories also have a limiting direction of the tangent at P_0 which, however, is different.
- The system (8) has an improper node at **0,** as its phase portrait Fig. 81 shows. The common limiting direction at **0** is that of the eigenvector $\mathbf{x}^{(1)} = [1 \quad 1]^T$ because e^{-4t} goes to zero faster than e^{-2t} as t increases. The two exceptional limiting tangent directions are those of $\mathbf{x}^{(2)} = [1 \quad -1]^T$ and $-\mathbf{x}^{(2)} = [-1 \quad 1]^T$.

continued

Fig. 81. Trajectories of the system (8) (Improper node)



EXAMPLE 2 Proper Node (Fig. 82)

- A **proper node** is a critical point $P0$ at which every trajectory has a definite limiting direction and for any given direction **d** at P_0 there is a trajectory having **d** as its limiting direction.

The system

$$(10) \qquad \mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_1 \\ y_2' &= y_2 \end{aligned}$$

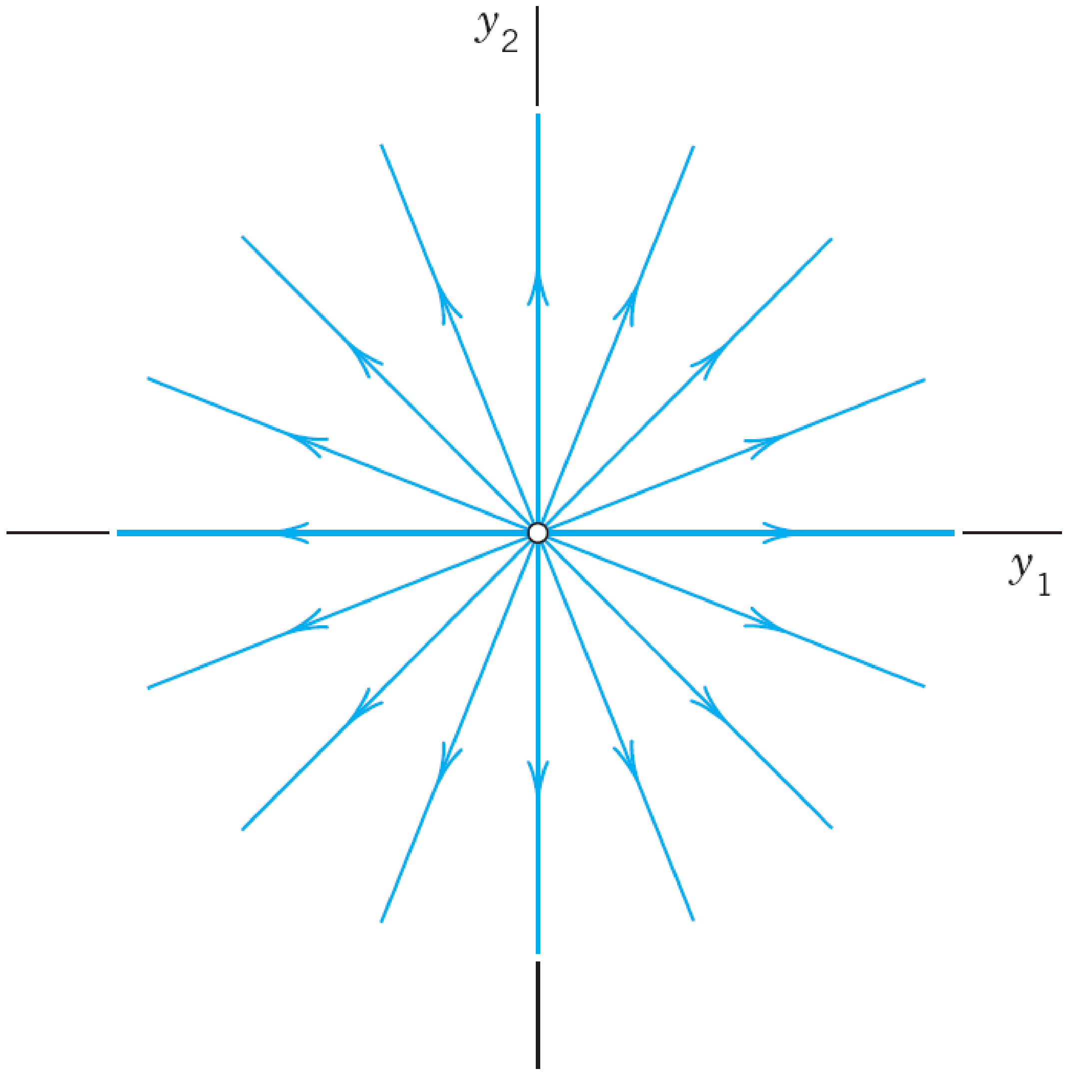
continued

has a proper node at the origin (see Fig. 82). Indeed, the matrix is the unit matrix. Its characteristic equation $(1 - \lambda)^2 = 0$ has the root $\lambda = 1$. Any $\mathbf{x} \neq \mathbf{0}$ is an eigenvector, and we can take $[1 \quad 0]^T$ and $[0 \quad 1]^T$. Hence a general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \qquad \text{or} \qquad \begin{aligned} y_1 &= c_1 e^t \\ y_2 &= c_2 e^t \end{aligned} \qquad \text{or} \qquad c_1 y_2 = c_2 y_1.$$

continued

Fig. 82. Trajectories of the system (10) (Proper node)



EXAMPLE3 Saddle Point (Fig. 83)

- A **saddle point** is a critical point P_0 at which there are two incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of P_0 bypass P_0.

The system

(11) $$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= \ \ y_1 \\ y_2' &= -y_2 \end{aligned}$$

continued

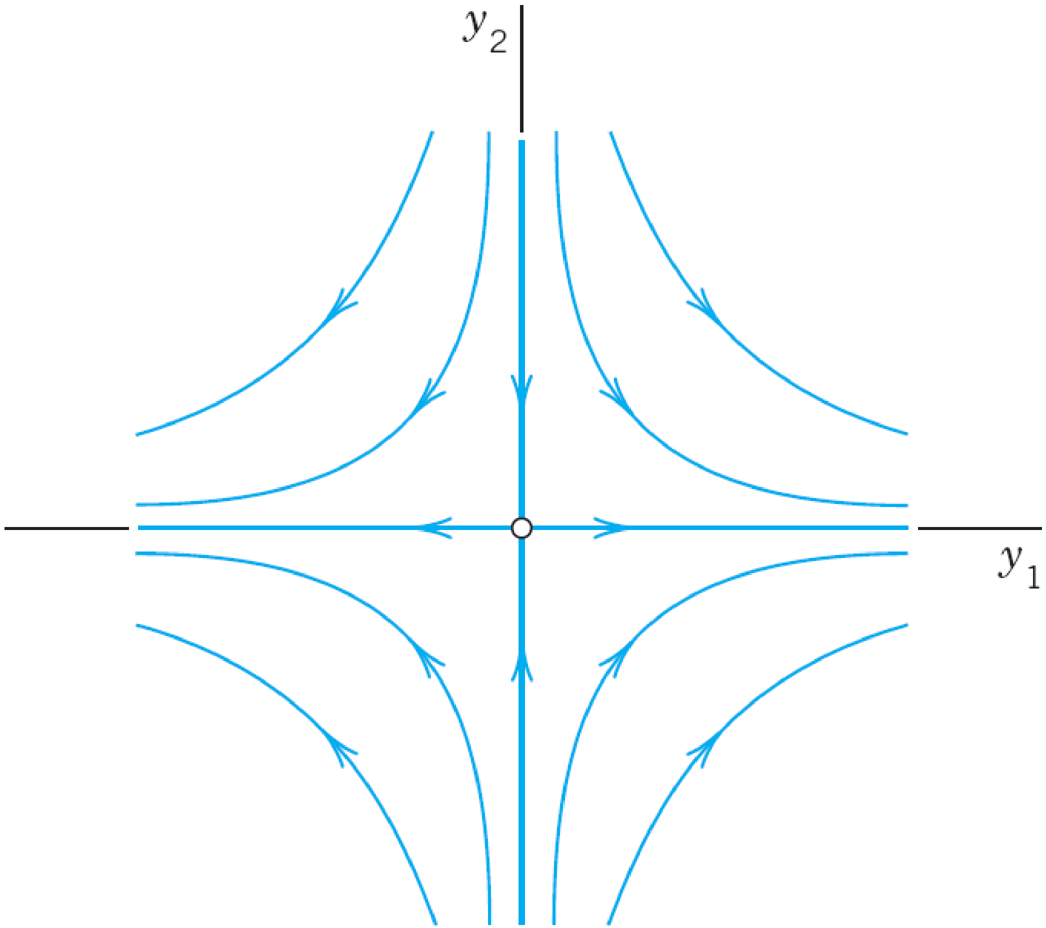
has a saddle point at the origin. Its characteristic equation $(1 - \lambda)(-1 - \lambda) = 0$ has the roots $\lambda_1 = 1$ and $\lambda_2 = -1$. For $\lambda_1 = 1$ an eigenvector $[1 \quad 0]^T$ is obtained from the second row of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, that is, $0x_1 + (-1 - 1)x_2 = 0$. For $\lambda_2 = -1$ the first row gives $[0 \quad 1]^T$. Hence a general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \quad \text{or} \quad \begin{aligned} y_1 &= c_1 e^t \\ y_2 &= c_2 e^{-t} \end{aligned} \quad \text{or} \quad y_1 y_2 = const.$$

- This is a family of hyperbolas (and the coordinate axes); see Fig. 83.

continued

Fig. 83. Trajectories of the system (11) (Saddle point)



EXAMPLE 4 Center (Fig. 84)

- A **center** is a critical point that is enclosed by infinitely many closed trajectories.

The system

(12) $$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= -4y_1 \end{aligned}$$

has a center at the origin. The characteristic equation $\lambda^2 + 4 = 0$ gives the eigenvalues $2i$ and $-2i$. For $2i$ an eigenvector follows from the first equation $-2ix_1 + x_2 = 0$ of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, say, $[1 \quad 2i]^T$. For $\lambda = -2i$ that equation is $-(-2i)x_1 + x_2 = 0$ and gives, say, $[1 \quad -2i]^T$. Hence a complex general solution is

(12*) $$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it}, \qquad \text{thus} \qquad \begin{aligned} y_1 &= c_1 e^{2it} + c_2 e^{-2it} \\ y_2 &= 2ic_1 e^{2it} - 2ic_2 e^{-2it}. \end{aligned}$$

continued

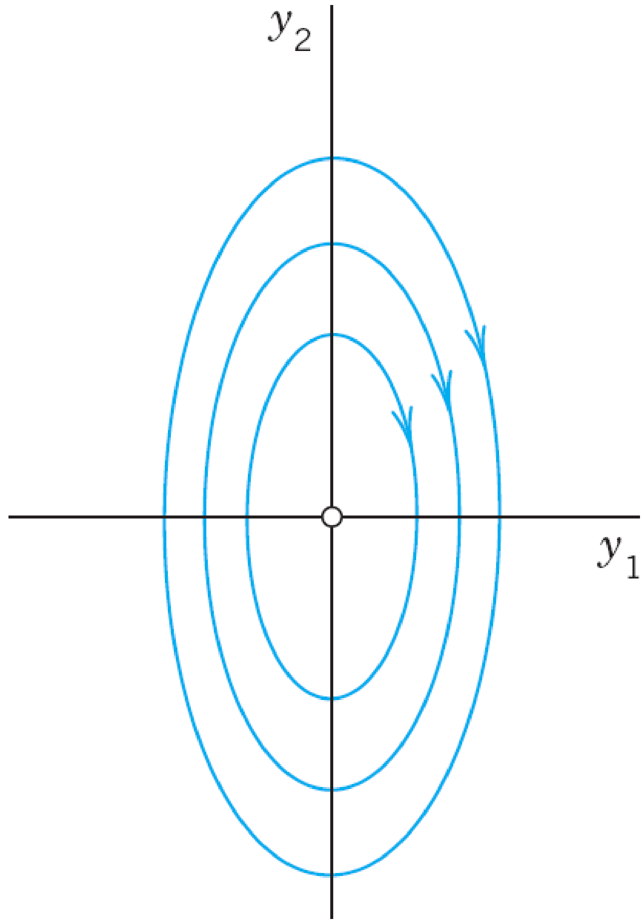
The next step would be the transformation of this solution to real form by the Euler formula (Sec. 2.2). But we were just curious to see what kind of eigenvalues we obtain in the case of a center. Accordingly, we do not continue, but start again from the beginning and use a shortcut. We rewrite the given equations in the form $y'_1 = y_2$, $4y_1 = -y'_2$; then the product of the left sides must equal the product of the right sides,

$$4y_1y_1' = -y_2y_2'. \qquad \text{By integration,} \qquad 2y_1^2 + \tfrac{1}{2}y_2^2 = const.$$

- This is a family of ellipses (see Fig. 84) enclosing the center at the origin.

continued

Fig. 84. Trajectories of the system (12) (Center)



EXAMPLE5 Spiral Point (Fig. 85)

- A **spiral point** is a critical point P_0 about which the trajectories spiral, approaching P_0 as $t \to \infty$(or tracing these spirals in the opposite sense, away from P_0).

The system

(13) $$\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned}$$

continued

has a spiral point at the origin, as we shall see. The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$. It gives the eigenvalues $-1 + i$ and $-1 - I$. Corresponding eigenvectors are obtained from $(-1 - \lambda)x_1 + x_2 = 0$. For $\lambda = -1 + i$ this becomes $-ix_1 + x_2 = 0$ and we can take $[1 \quad i]^T$ as an eigenvector. Similarly, an eigenvector corresponding to $-1 - i$ is $[1 \quad -i]^T$. This gives the complex general solution

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}.$$

continued

The next step would be the transformation of this complex solution to a real general solution by the Euler formula. But, as in the last example, we just wanted to see what eigenvalues to expect in the case of a spiral point. Accordingly, we start again from the beginning and instead of that rather lengthy systematic calculation we use a shortcut. We multiply the first equation in (13) by y_1, the second by y_2, and add, obtaining

$$y_1 y'_1 + y_2 y'_2 = -(y_1^2 + y_2^2).$$

continued

We now introduce polar coordinates r, t, where $r^2 = y_1^2 + y_2^2$. Differentiating this with respect to t gives $2rr' = 2y_1y'_1 + 2y_2y'_2$. Hence the previous equation can be written

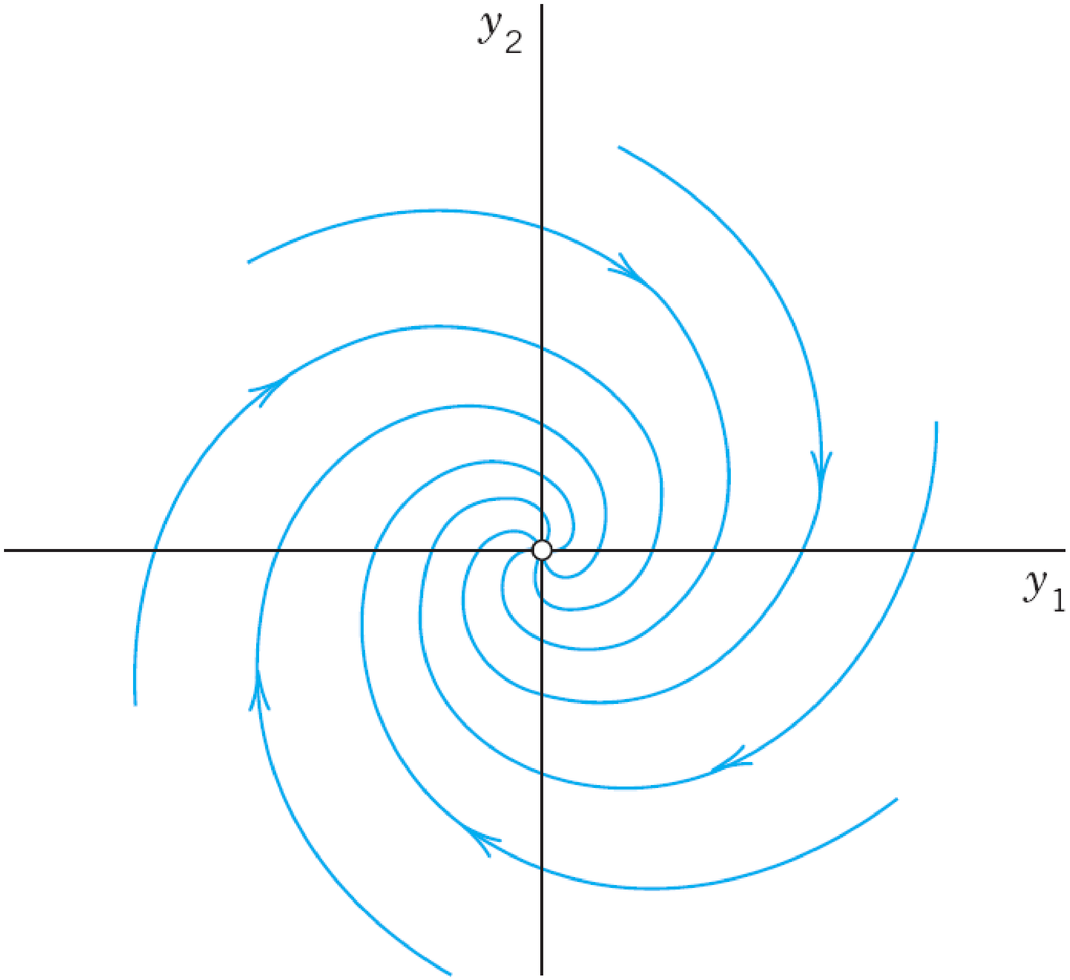
$$rr' = -r^2,$$

Thus, $r' = -r$, $dr/r = -dt$, $\ln|r| = -t + c^*$, $r = ce^{-t}$.

- For each real c this is a spiral, as claimed. (see Fig. 85).

continued

Fig. 85. Trajectories of the system (13) (Spiral point)



EXAMPLE 6 No Basis of Eigenvectors Available. Degenerate Node (Fig. 86)

- This cannot happen if **A** in (1) is symmetric ($a_{kj} = a_{jk}$, as in Examples 1–3) or skew-symmetric ($a_{kj} = -a_{jk}$, thus $a_{jj} = 0$). And it does not happen in many other cases (see Examples 4 and 5). Hence it suffices to explain the method to be used by an example.

Find and graph a general solution of

(14) $$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}.$$

continued

- ***Solution.*** **A** is not skew-symmetric! Its characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0.$$

It has a double root $\lambda = 3$. Hence eigenvectors are obtained from $(4 - \lambda)x_1 + x_2 = 0$, thus from $x_1 + x_2 = 0$, say, $\mathbf{x}^{(1)} = [1 \quad -1]^T$ and nonzero multiples of it (which do not help). The method now is to substitute

$$\mathbf{y}^{(2)} = \mathbf{x}te^{\lambda t} + \mathbf{u}e^{\lambda t}$$

continued

with constant $\mathbf{u} = [u_1 \quad u_2]^T$ into (14). (The $\mathbf{x}t$-term alone, the analog of what we did in Sec. 2.2 in the case of a double root, would not be enough. Try it.) This gives

$$y'^{(2)} = \mathbf{x}e^{\lambda t} + \lambda \mathbf{x}te^{\lambda t} + \lambda \mathbf{u}e^{\lambda t} = \mathbf{A}\mathbf{y}^{(2)} = \mathbf{A}\mathbf{x}te^{\lambda t} + \mathbf{A}\mathbf{u}e^{\lambda t}.$$

On the right, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Hence the terms $\lambda\mathbf{x}te^{\lambda t}$ cancel, and then division by $e^{\lambda t}$ gives

$$\mathbf{x} + \lambda\mathbf{u} = \mathbf{A}\mathbf{u}, \text{ thus } (\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{x}.$$

Here $\lambda = 3$ and $\mathbf{x} = [1 \quad -1]^T$, so that

$$(\mathbf{A} - 3\mathbf{I})\mathbf{u} = \begin{bmatrix} 4-3 & 1 \\ -1 & 2-3 \end{bmatrix}\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \text{thus} \qquad \begin{aligned} u_1 + u_2 &= 1 \\ -u_1 - u_2 &= -1. \end{aligned}$$

continued

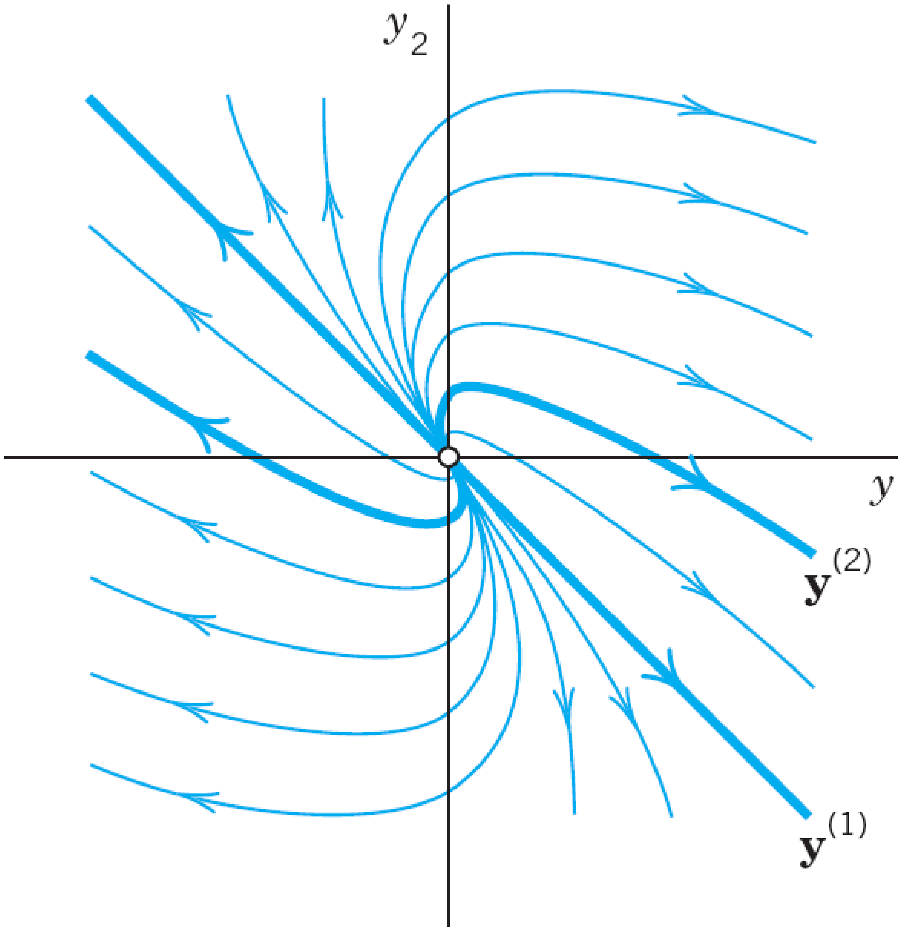
A solution, linearly independent of $\mathbf{x} = [1 \quad -1]^T$, is $\mathbf{u} = [0 \quad 1]^T$. This yields the answer (Fig. 86)

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) e^{3t}.$$

- The critical point at the origin is often called a **degenerate node.** $c_1\mathbf{y}^{(1)}$ gives the heavy straight line, with $c_1 > 0$ the lower part and $c_1 < 0$ the upper part of it. $\mathbf{y}^{(2)}$ gives the right part of the heavy curve from 0 through the second, first, and—finally—fourth quadrants. $-\mathbf{y}^{(2)}$ gives the other part of that curve.

continued

Fig. 86. Degenerate node in Example 6



4.4 Criteria for Critical Points. Stability

- We continue our discussion of homogeneous linear systems with constant coefficients

(1) $\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{y},$ in components, $\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned}$

(3)

$$\frac{dy_2}{dy_1} = \frac{y_2' \, dt}{y_1' \, dt} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2} .$$

We also recall from Sec. 4.3 that there are various types of critical points, and we shall now see how these types are related to the eigenvalues. The latter are solutions $\lambda=\lambda_1$ and λ_2 of the characteristic equation

(4)

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \det \mathbf{A} = 0.$$

continued

This is a quadratic equation $\lambda^2 - p\lambda + q = 0$ with coefficients p, q and discriminant Δ given by

(5) $p = a_{11} + a_{22}$, $q = \det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$,

$$\Delta = p^2 - 4q.$$

From calculus we know that the solutions of this equation are

(6) $$\lambda_1 = \tfrac{1}{2}(p + \sqrt{\Delta}), \qquad \lambda_2 = \tfrac{1}{2}(p - \sqrt{\Delta}).$$

Table 4.1 Eigenvalue Criteria for Critical Points (Derivation after Table 4.2)

Name	$p = \lambda_1 + \lambda_2$	$q = \lambda_1 \lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	Comments on λ_1, λ_2
(a) Node		$q > 0$	$\Delta \geqq 0$	Real, same sign
(b) Saddle point		$q < 0$		Real, opposite sign
(c) Center	$p = 0$	$q > 0$		Pure imaginary
(d) Spiral point	$p \neq 0$		$\Delta < 0$	Complex, not pure imaginary

Stability

DEFINITION

Stable, Unstable, Stable and Attractive(1)

A critical point P_0 of (1) is called stable if, roughly, all trajectories of (1) that at some instant are close to P_0 remain close to P_0 at all future times; precisely: if for every disk D_ε of radius $\varepsilon < 0$ with center P_0 there is a disk D_δ of radius $\delta > 0$ with center P_0 such that every trajectory of (1) that has a point P_1 (corresponding to $t = t_1$, say) in $D\delta$ has all its points corresponding to $t \geq t_1$ in D_ε. See Fig. 89.

continued

DEFINITION

Stable, Unstable, Stable and Attractive(2)

P_0 is called unstable if P_0 is not stable.

P_0 is called stable and attractive (or asymptotically stable) if P_0 is stable and every trajectory that has a point in D_δ approaches P_0 as $t \rightarrow \infty$. See Fig. 90.

Fig. 89. Stable critical point P_0 of (1) (The trajectory initiating at P_1 stays in the disk of radius ε.)

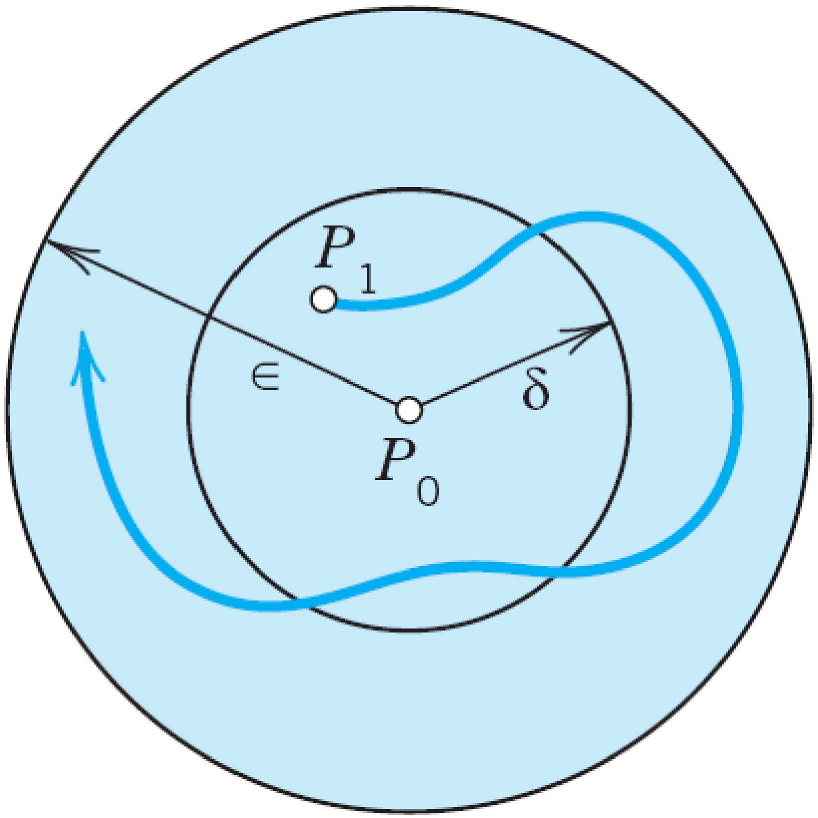


Fig. 90. Stable and attractive critical point *P*0 of (1)

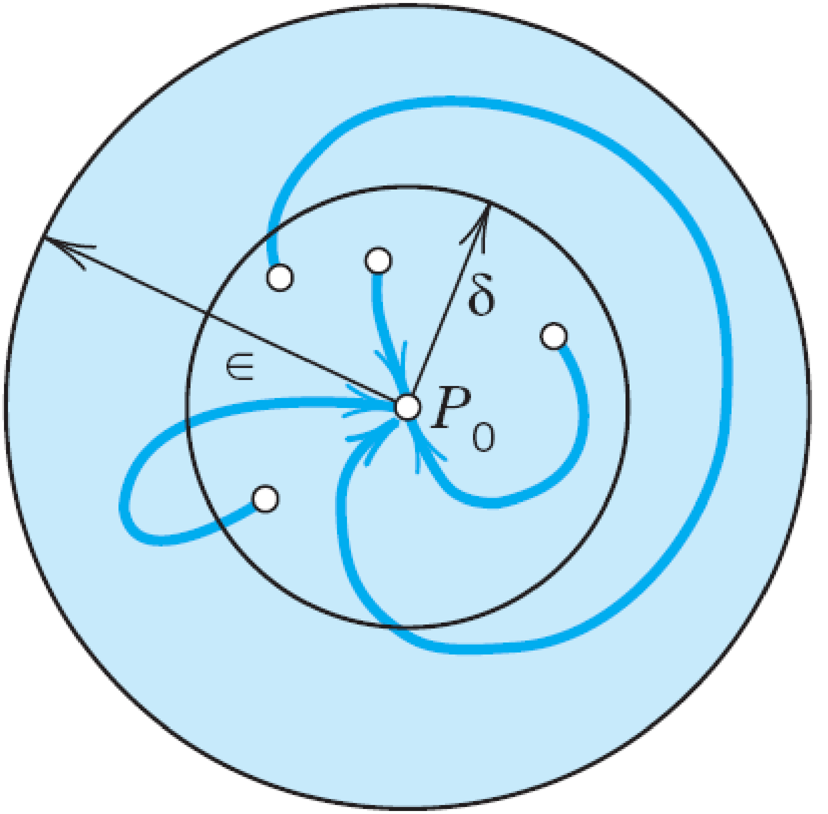


Table 4.2 Stability Criteria for Critical Points

Type of Stability	$p = \lambda_1 + \lambda_2$		$q = \lambda_1 \lambda_2$
(a) Stable and attractive	$p < 0$		$q > 0$
(b) Stable	$p \leqq 0$		$q > 0$
(c) Unstable	$p > 0$	OR	$q < 0$

EXAMPLE 1 Application of the Criteria in Tables 4.1 and 4.2

- In Example 1, Sec. 4.3, we have $\mathbf{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}$, $p = -6$, $q = 8$, $\Delta = 4$, a node by Table 4.1(a), which is stable and attractive by Table 4.2(a).

EXAMPLE 2 Free Motions of a Mass on a Spring

- What kind of critical point does $my'' + cy' + ky = 0$ in Sec. 2.4 have?
- ***Solution.*** Division by m gives $y'' = -(k/m)y - (c/m)y'$. To get a system, set $y_1 = y$, $y_2 = y'$ (see Sec. 4.1). Then $y'_2 = y'' = -(k/m)y_1 - (c/m)y_2$. Hence

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \mathbf{y}, \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -k/m & -c/m - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

continued

We see that $p = -c/m$, $q = k/m$, $\Delta = (c/m)^2 - 4k/m$. From this and Tables 4.1 and 4.2 we obtain the following results. Note that in the last three cases the discriminant Δ plays an essential role.

- ***No damping.*** $c = 0$, $p = 0$, $q > 0$, a center.
- ***Underdamping.*** $c^2 < 4mk$, $p < 0$, $q > 0$, $\Delta < 0$, a stable and attractive spiral point.
- ***Critical damping.*** $c^2 = 4mk$, $p < 0$, $q > 0$, $\Delta = 0$, a stable and attractive node.
- ***Overdamping.*** $c^2 > 4mk$, $p < 0$, $q > 0$, $\Delta > 0$, a stable and attractive node.

4.6 Nonhomogeneous Linear Systems of ODEs

In this last section of Chap. 4 we discuss methods for solving nonhomogeneous linear systems of ODEs

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \qquad (1)$$

where the vector $\mathbf{g}(t)$ is not identically zero.

We assume $\mathbf{g}(t)$ and the entries of the $n \times n$ matrix $\mathbf{A}(t)$ to be continuous on some interval J of the t-axis.

continued

From a general solution $\mathbf{y}^{(h)}(t)$ of the homogeneous system

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

and a **particular solution** $\mathbf{y}^{(p)}(t)$ of (1), we get

$$\mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)}. \tag{2}$$

where **y** is called a **general solution** of (1) because it includes every solution of (1).

Method of Undetermined Coefficients

As for a single ODE, this method is suitable if (1) the entries of **A** are constants and (2) the components of **g** are

- constants,
- positive integer powers of *t*,
- exponential functions, or
- cosines and sines.

In such a case a particular solution $\mathbf{y}^{(p)}$ is assumed in a form similar to **g;** for instance, $\mathbf{y}^{(p)} = \mathbf{u} + \mathbf{v}t + \mathbf{w}t^2$ if **g** has components quadratic in *t*, with **u, v, w** to be determined by substitution into (1). **This is similar to Sec. 2.7, except for the Modification Rule.** .

Example 1 Method of Undetermined Coefficients. Modification Rule

Find a general solution of

(3)
$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}.$$

Solution. A general equation of the homogeneous system is (see Example 1 in Sec. 4.3)

(4)
$$\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}.$$

continued

Since λ = –2 is an eigenvalue of **A**, the function e^{-2t} on the right also appears in $\mathbf{y}^{(h)}$. Thus, we must apply the *Modification Rule* by setting

$$\mathbf{y}^{(p)} = \mathbf{u}te^{-2t} + \mathbf{v}e^{-2t}$$

Note that the first of these two terms is the analog of the modification in Sec. 2.7, but it would not be sufficient here. **(Try it.)**

By substitution,

$$\frac{d}{dt}\mathbf{y}^{(p)} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} - 2\mathbf{v}e^{-2t} = \mathbf{A}\mathbf{u}te^{-2t} + \mathbf{A}\mathbf{v}e^{-2t} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

continued

Equating the te^{-2t}-terms on both sides, we have

$$-2\mathbf{u} = \mathbf{A}\mathbf{u}$$

Hence **u** must be an eigenvector of **A** corresponding to $\lambda = -2$; thus $\mathbf{u} = a[1 \quad 1]^T$ with any $a \neq 0$.

Equating the e^{-2t}-terms gives

$$\mathbf{u} - 2\mathbf{v} = \mathbf{A}\mathbf{v} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} \qquad \text{thus} \qquad \begin{bmatrix} a \\ a \end{bmatrix} - \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} = \begin{bmatrix} -3v_1 + v_2 \\ v_1 - 3v_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}.$$

Collecting terms and reshuffling gives

$$v_1 - v_2 = -a - 6$$

$$-v_1 + v_2 = -a + 2.$$

continued

By addition, $0 = -2a - 4$, $a = -2$, $\therefore v_2 = v_1 + 4$. Let $v_1 = k$

$$\mathbf{v} = [k \quad k + 4]^{\mathrm{T}}$$

We can simply choose $k = 0$. This gives

$$\mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} e^{-2t}.$$

$$\Rightarrow \mathbf{y}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} k \\ k+4 \end{bmatrix} e^{-2t}$$

$$= \tilde{c}_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} e^{-2t}$$

where $\tilde{c}_1 = c_1 + k$.

Method of Variation of Parameters

- This method can be applied to nonhomogeneous linear systems

(6) $$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{g}(t)$$

with variable $\mathbf{A} = \mathbf{A}(t)$ and general $\mathbf{g}(t)$. It yields a particular solution $\mathbf{y}^{(p)}$ of (6) on some open interval J on the t-axis if a general solution of the homogeneous system $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ on J is known. We explain the method in terms of the previous example.

EXAMPLE 2 Solution by the Method of Variation of Parameters

- Solve (3) in Example 1.
- ***Solution.*** A basis of solutions of the homogeneous system is $[e^{-2t} \quad e^{-2t}]^T$ and $[e^{-4t} \quad e^{-4t}]^T$. Hence the general solution (4) of the homogenous system may be written

$$\mathbf{y}^{(h)} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{Y}(t)\mathbf{c}. \tag{7}$$

continued

Here, $\mathbf{Y}(t) = [\mathbf{y}^{(1)} \quad \mathbf{y}^{(2)}]^T$ is the fundamental matrix (see Sec. 4.2). As in Sec. 2.10 we replace the constant vector $\mathbf{c}$ by a variable vector $\mathbf{u}(t)$ to obtain a particular solution

$$\mathbf{y}^{(p)} = \mathbf{Y}(t)\mathbf{u}(t).$$

Substitution into (3) $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$ gives

$$\mathbf{Y}'\mathbf{u} + \mathbf{Y}\mathbf{u}' = \mathbf{A}\mathbf{Y}\mathbf{u} + \mathbf{g}. \tag{8}$$

Now since $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are solutions of the homogeneous system, we have

$$\mathbf{y}^{(1)\prime} = \mathbf{A}\mathbf{y}^{(1)}, \quad \mathbf{y}^{(2)\prime} = \mathbf{A}\mathbf{y}^{(2)}, \quad \text{thus } \mathbf{Y}' = \mathbf{A}\mathbf{Y}.$$

continued

Hence **Y'u** = **AYu,** so that (8) reduces to

Yu' = **g.** The solution is **u'** = Y^{-1}g**;**

here we use that the inverse Y^{-1} of **Y** (Sec. 4.0) exists because the determinant of **Y** is the Wronskian *W*, which is not zero for a basis. Equation (9) in Sec. 4.0 gives the form of Y^{-1},

$$\mathbf{Y}^{-1} = \frac{1}{-2e^{-6t}} \begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}.$$

continued

We multiply this by **g,** obtaining

$$\mathbf{u}' = \mathbf{Y}^{-1}\mathbf{g} = \frac{1}{2}\begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}\begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 \\ -8e^{2t} \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}.$$

Integration is done componentwise (just as differentiation) and gives

$$\mathbf{u}(t) = \int_0^t \begin{bmatrix} -2 \\ -4e^{2\tilde{t}} \end{bmatrix} d\tilde{t} = \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

continued

(where +2 comes from the lower limit of integration). From this and **Y** in (7) we obtain

$$\mathbf{Yu} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix} = \begin{bmatrix} -2te^{-2t} - 2e^{-2t} + 2e^{-4t} \\ -2te^{-2t} + 2e^{-2t} - 2e^{-4t} \end{bmatrix} = \begin{bmatrix} -2t - 2 \\ -2t + 2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-4t}.$$

The last term on the right is a solution of the homogeneous system. Hence we can absorb it into $\mathbf{y}^{(h)}$. We thus obtain as a general solution of the system (3), in agreement with (5*),

$$\text{(9)} \qquad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}.$$

SUMMARY OF CHAPTER 4

- Whereas single electric circuits or single mass–spring systems are modeled by single ODEs (Chap. 2), networks of several circuits, systems of several masses and springs, and other engineering problems lead to **systems of ODEs,** involving several unknown functions $y_1(t)$, ..., $y_n(t)$. Of central interest are **first-order systems** (Sec. 4.2):

$\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$, in components,

$$\begin{aligned} y_1' &= f_1(t, y_1, \cdots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, \cdots, y_n), \end{aligned}$$

continued

to which higher order ODEs and systems of ODEs can be reduced (Sec. 4.1). In this summary we let $n = 2$, so that

(1) $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$, in components,
$$y'_1 = f_1(t, y_1, y_2)$$
$$y'_2 = f_2(t, y_1, y_2)$$

continued

- Then we can represent solution curves as **trajectories** in the **phase plane** (the y_1y_2-plane), investigate their totality [the **“phase portrait”** of (1)], and study the kind and **stability** of the **critical points** (points at which both f_1 and f_2 are zero), and classify them as **nodes, saddle points, centers,** or **spiral points** (Secs. 4.3, 4.4). These phase plane methods are **qualitative;** with their use we can discover various general properties of solutions without actually solving the system. They are primarily used for **autonomous systems,** that is, systems in which t does not occur explicitly.

continued

- A **linear system** is of the form

(2) $$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

If **g = 0,** the system is called **homogeneous** and is of the form

(3) $$\mathbf{y}' = \mathbf{A}\mathbf{y}.$$

continued

If a_{11}, ..., a_{22} are constants, it has solutions $\mathbf{y} = \mathbf{x}e^{\lambda t}$, where λ is a solution of the quadratic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

and $\mathbf{x} \neq \mathbf{0}$ has components x_1, x_2 determined up to a multiplicative constant by

$$(a_{11} - \lambda)x_1 + a_{12}x_2 = 0.$$

(These λ's are called the **eigenvalues** and these vectors **x eigenvectors** of the matrix **A.** Further explanation is given in Sec. 4.0.)

continued

- A system (2) with $\mathbf{g} \neq \mathbf{0}$ is called **nonhomogeneous.** Its general solution is of the form $\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$, where $\mathbf{y}_h$ is a general solution of (3) and $\mathbf{y}_p$ a particular solution of (2). Methods of determining the latter are discussed in Sec. 4.6.
- The discussion of critical points of linear systems based on eigenvalues is summarized in Tables 4.1 and 4.2 in Sec. 4.4. It also applies to nonlinear systems if the latter are first linearized. The key theorem for this is Theorem 1 in Sec. 4.5, which also includes three famous applications, namely the pendulum and van der Pol equations and the Lotka–Volterra predator–prey population model.